

### ↳ Mathematical definitions of $R$ 's basic properties on $A$

Mathematical definitions of the basic properties of the relation  $R$  on the set  $A$ :

Let  $R$  be a relation on  $A$ , then

(1)  $R$  is *reflexive* on  $A$  if and only if  $I_A \subseteq R$

$R$  at least contains all reflexive pairs  $(x,x)$

(2)  $R$  is *irreflexive* on  $A$  if and only if  $R \cap I_A = \emptyset$

$R$  does not contain any  $(x,x)$

(3)  $R$  is *symmetric* on  $A$  if and only if  $R = R^{-1}$

(4)  $R$  is *antisymmetric* on  $A$  if and only if  $R \cap R^{-1} \subseteq I_A$

$R$  and its inverse relation contains only reflexive pairs  $(x,x)$

(5)  $R$  is *transitive* on  $A$  if and only if  $R \circ R \subseteq R$

### ↳ Proof of Reflexivity of $R$ on $A$

- To prove that  $R$  is reflexive on  $A$ :

- Proof Pattern:

For any  $x$ ,

$x \in A \Rightarrow$	.....	.....	.....	$\Rightarrow \langle x, x \rangle \in R$
Assume		reasoning process		Conclusion

*e.g.* >>> Example: Prove that if  $I_A \subseteq R$ , then  $R$  is reflexive on  $A$ .

Proof: For any  $x$ ,

$$x \in A \Rightarrow \langle x, x \rangle \in I_A \Rightarrow \langle x, x \rangle \in R$$

Therefore,  $R$  is reflexive on  $A$ .

### ↳ Proof of symmetric of $R$ on $A$

- To prove that  $R$  is symmetric on  $A$ :

- Proof Pattern:

For any  $\langle x, y \rangle$

$\langle x, y \rangle \in R \Rightarrow \dots \dots \dots \Rightarrow \langle y, x \rangle \in R$   
 Assumption      Reasoning process      Conclusion

*e.g.* >>> Example: Prove that if  $R=R^{-1}$ , then  $R$  is symmetric on  $A$ .

Proof: Let  $\langle x, y \rangle$

$\langle x, y \rangle \in R \Rightarrow \langle y, x \rangle \in R^{-1} \Rightarrow \langle y, x \rangle \in R$

Therefore,  $R$  is symmetric on  $A$ .

### ↳ Proof of antisymmetric of $R$ on $A$

- To prove that  $R$  is antisymmetric on  $A$ :

- Proof Pattern:

For any  $\langle x, y \rangle$

$\langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \Rightarrow \dots \dots \dots \Rightarrow x=y$

Assumption

Reasoning process

Conclusion

*e.g.* >>> Example: Prove that if  $R \cap R^{-1} \subseteq I_A$ , then  $R$  is antisymmetric on  $A$ .

Proof: For any  $\langle x, y \rangle$

$\langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \Rightarrow \langle x, y \rangle \in R \wedge \langle x, y \rangle \in R^{-1}$

$\Rightarrow \langle x, y \rangle \in R \cap R^{-1} \Rightarrow \langle x, y \rangle \in I_A \Rightarrow x=y$

Therefore,  $R$  is antisymmetric on  $A$ .

### ↳ Proof of transitive of $R$ on $A$

- To prove that  $R$  is transitive on  $A$ :

- Proof Pattern:

For any  $\langle x, y \rangle, \langle y, z \rangle$

$$\langle x, y \rangle \in R \wedge \langle y, z \rangle \in R \Rightarrow \dots \Rightarrow \langle x, z \rangle \in R$$

Assumption Reasoning process Conclusion

*e.g.* >>> Example 7: Prove that if  $R \circ R \subseteq R$ , then  $R$  is transitive on  $A$ .

Proof: Let  $\langle x, y \rangle, \langle y, z \rangle$

$$\langle x, y \rangle \in R \wedge \langle y, z \rangle \in R \Rightarrow \langle x, z \rangle \in R \circ R \Rightarrow \langle x, z \rangle \in R$$

Therefore,  $R$  is transitive on  $A$ .

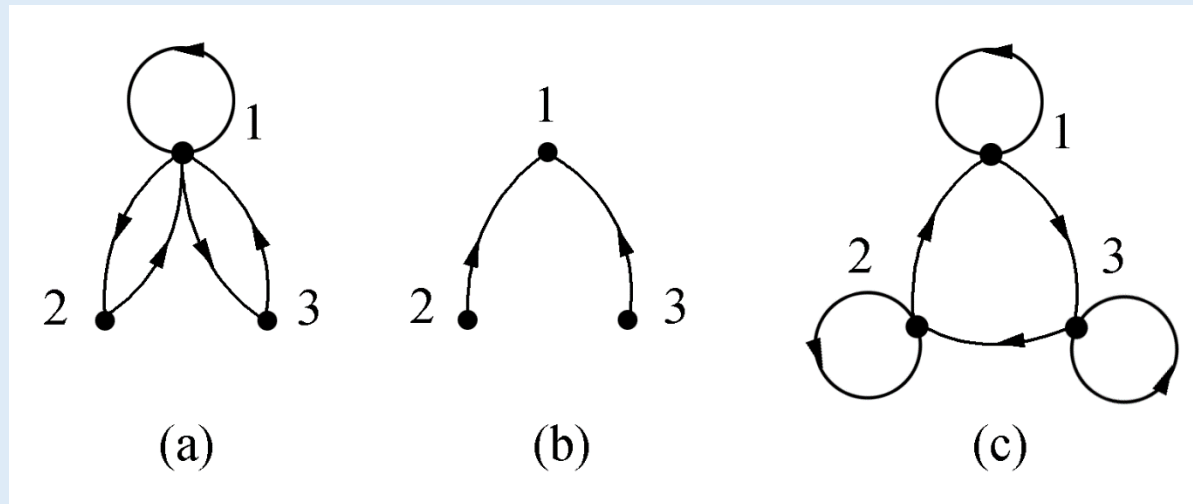
## ↳ Comparison Table of Properties of Relation $R$

Relation Properties	Expression	Definition	Relation Matrix	Relation Diagram
Reflexivity	$I_A \subseteq R$	$\forall x \in A, \exists \langle x, x \rangle \in R$	Main diagonal elements are 1	Every vertex has a loop
Irreflexivity	$R \cap I_A = \emptyset$	$\forall x \in A, \exists \langle x, x \rangle \notin R$	Main diagonal elements are 0	No loops at any vertex
Symmetry	$R = R^{-1}$	If $\langle x, y \rangle \in R$ , then $\langle y, x \rangle \in R$	The matrix is a symmetric matrix	If there is an edge between two vertices, it must be a bidirectional edges.

Relation Properties	Expression	Definition	Relation Matrix	Relation Diagram
Antisymmetry	$R \cap R^{-1} \subseteq I_A$	If Expression $\langle x, y \rangle \in R$ and $x \neq y$ , then $\langle y, x \rangle \notin R$	If $r_{ij} = 1$ , and $i \neq j$ , Then $r_{ji} = 0$	If there is an edge between two points, it must be a directed edge (no bidirectional edges)
Transitivity	$R \circ R \subseteq R$	If $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ , then $\langle x, z \rangle \in R$	If $M_{ij}^2 = 1$ , $M_{ij} = 1$	If there is an edge from vertex $x_i$ to $x_j$ , and an edge from $x_j$ to $x_k$ , then there is also an edge from $x_i$ to $x_k$ .

## ↳ Determine the Relation Based on the Relation Diagram (e.g.)

e.g. >>> **Example 8:** Determine the properties of the relationship in the figure below and explain the reasoning.



- (a) Neither reflexive nor antireflexive; symmetric, not antisymmetric; not transitive.
- (b) Antireflexive, not reflexive; antisymmetric, not symmetric; transitive.
- (c) Reflexive, not antireflexive; antisymmetric, not symmetric; not transitive.

	Reflexivity	Irreflexivity	Symmetry	Antisymmetry	Transitivity
$R_1^{-1}$	√	√	√	√	√
$R_1 \cap R_2$	√	√	√	√	√
$R_1 \cup R_2$	√	√	√	×	×
$R_1 - R_2$	×	√	√	√	×
$R_1 \circ R_2$	√	×	×	×	×

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- 4.3.1 Definition and Determination of Relation Properties
  - Reflexivity and Irreflexivity
  - Symmetry and Antisymmetry
  - Transitivity
- 4.3.2 Closure of Relations
  - Definition of Closure
  - Closure Calculation
  - Warshall's Algorithm